Midterm 1 Review:

Higher priority

Matrix multiplication

Find matrix inverses

1.1: 17, 18, 20, 36

1.2: 6, 38

1.3: 9, 16, 17, 18, 28, 36

2.1: 4, 5, 6

2.2: 1, 4, 6, 10, 14

2.3: 2, 6, 8, 10, 11, 19, 20

2.4: 2, 4, 5, 16, 17, 30, 38, 76

3.1: 4, 44

Quiz 2:

Is $T\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}x_1 + 2x_2 + \boxed{2}\\2x_1 + x_2 + \boxed{3}\end{bmatrix}$ linear?

$\boxed{\text{No}}$, it's not linear

① $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$

$T\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

~~②~~ $T(c\vec{x}) = cT(\vec{x})$

$T(\vec{0}) = \vec{0}$

$$\boxed{T\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0+0+2\\0+0+3\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix}}$$

this invalidates condition ②

What makes a transformation not linear

① Existence of powers higher than 2 or cross-terms (x_1x_2, x_3x_4)

② Existence of constants (deg 0 terms)

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ 2x_1 + x_2 \end{bmatrix}$$, show this is linear by verifying conditions ① + ② above

① $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$$T(\vec{x}+\vec{y}) = T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} x_1 + y_1 + 2(x_2 + y_2) \\ 2(x_1 + y_1) + x_2 + y_2 \end{bmatrix}$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ 2x_1 + x_2 \end{bmatrix} + T\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} y_1 + 2y_2 \\ 2y_1 + y_2 \end{bmatrix}$$

$$T(\vec{x}) + T(\vec{y}) = \begin{bmatrix} x_1 + 2x_2 \\ 2x_1 + x_2 \end{bmatrix} + \begin{bmatrix} y_1 + 2y_2 \\ 2y_1 + y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 + 2x_2 + 2y_2 \\ 2x_1 + 2y_1 + x_2 + y_2 \end{bmatrix}$$

② $T(c\vec{x}) = c\,T(\vec{x})$

$$T\left(c\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}\right) = \begin{bmatrix} cx_1 + 2cx_2 \\ 2cx_1 + cx_2 \end{bmatrix} = c\begin{bmatrix} x_1 + 2x_2 \\ 2x_1 + x_2 \end{bmatrix}$$

$$= c\,T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) \checkmark$$

Quiz 3:

"divide by the matrix" $\qquad x^2 = x \cdot 1,\ x \neq 0$

If A is a 2×2 invertible matrix w/ $\boxed{A^2 = A}$, then $A = I_2$.

$$A^{-1}(A^2) = \boxed{A^{-1}A} = I_2$$

$$\cancel{A^{-1}A} \cdot A = A^{-1}A$$

$$I_2 \cdot A = I_2$$

$$A = I_2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

If A 2×2 w/ $A^2 = I_2$, then $A = I_2$ or $A = -I_2$. $\boxed{\text{False}}$

$$A \cdot \boxed{A} = I_2$$

$$x^2 = 1 \Rightarrow x = \pm 1$$

ex: $\boxed{A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + 0\cdot 0 & 1\cdot 0 + 0\cdot(-1) \\ 0\cdot 1 + 0\cdot(-1) & 0\cdot 0 + (-1)\cdot(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$A^2 = I_2$, $A^{-1} \cdot A^2 = A^{-1} \cdot I_2$

$$\boxed{A = A^{-1}}$$

(31) There exists a 4×3 matrix A of $\boxed{\text{rank } 3}$ such that

$$\boxed{A\begin{bmatrix}1\\2\\3\end{bmatrix}=\vec{0}}$$

$$\downarrow$$

$$\boxed{\vec{a}_1+2\vec{a}_2+3\vec{a}_3=\vec{0}}$$

$$\vec{a}_1=-2\vec{a}_2-3\vec{a}_3$$

$$\begin{array}{cc} 4 & \begin{bmatrix} | & | & | \\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \\ | & | & | \end{bmatrix} \\ & 3 \end{array}$$

rank 3 $\downarrow$ 3 pivots

$\boxed{3.1}$

(44) A, $B = \text{rref}(A)$

a. Is $\ker(A) = \ker(B)$?

b. Is $\text{im}(A) = \text{im}(B)$?

A, $m \times n$, $A = \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix}$

$$\text{im}(A) = \text{span}\{\vec{a}_1, \ldots, \vec{a}_n\}$$

$$= \left\{ \begin{array}{l} \text{all linear combinations of} \\ \text{the vectors } \vec{a}_1, \ldots, \vec{a}_n \end{array} \right\}$$

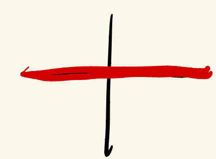
$$= \left\{ \begin{array}{l} c_1\vec{a}_1 + \cdots + c_n\vec{a}_n \\ c_1, \ldots, c_n \text{ scalars} \end{array} \right\}$$

ex: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$$\text{im}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cancel{c_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix}} \right\} = \left\{ \begin{bmatrix} c_1 \\ 0 \end{bmatrix} \right\}$$

c_1 scalar



$$\text{im}(A) = \left\{ \begin{array}{l} \text{all vectors } \vec{b} \\ \text{s.t. } A\vec{x} = \vec{b} \text{ has a solution} \end{array} \right\}$$

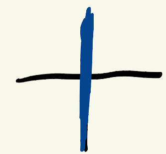
$$\ker(A) = \left\{ \text{all solutions to } A\vec{x} = \vec{0} \right\}$$

$$\underset{A}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}} \Rightarrow \boxed{x_1 = 0}$$

$$\Rightarrow \ker(A) = \left\{ \begin{bmatrix} 0 \\ x_2 \end{bmatrix} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



$\boxed{3.1}$

⑥ $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, find basis for $\boxed{\ker(A)}$

$A\vec{x} = \vec{0}$

$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right] -(I) \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] -(II)$$

$$\rightarrow \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \end{array} \Rightarrow \begin{array}{l} x_1 - x_3 = 0 \\ x_2 - 2x_3 = 0 \end{array} \quad \begin{array}{l} x_1 = x_3 \\ x_2 = -2x_3 \end{array}$$

$\downarrow$

x_3 free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = \boxed{x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}$$

$$\Rightarrow \ker(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\} =$$

basis: minimal list of spanning vectors

$$\ker(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \underbrace{\begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix}}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}} \right\}$$

Steps to find basis for ker(A):

1. Augment matrix w/ $\vec{0}$
2. Row reduce & identify which cols. <u>don't</u> have pivots
   ↳ the corresp. variables are free & corresp. to a vector in the basis
3. Turn rows of matrix back into linear equations
4. Solve for pivot variables in terms of free variables

5. Plug that back into vector $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

6. Pull out the free variables to find basis vector(s)

$\boxed{3.1}$

44. A, $B = rref(A)$

a. Is $\ker(A) = \ker(B)$?

b. Is $im(A) = im(B)$?

a. Yes, row reduction does not change solution set to $A\vec{x} = \vec{0}$

b. No,

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{-(I)} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$im = span\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ $im = span\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

$$\begin{array}{c c} & \begin{array}{cccc} & & 4 & \end{array} \\ 2 & \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

A

x_3 x_4 free variables

$$x_1 - x_3 = 0$$

$$x_2 - x_4 = 0$$

$$\boxed{\begin{aligned} x_1 &= x_3 \\ x_2 &= x_4 \end{aligned}}$$

$$\begin{bmatrix} x_3 \\ x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ 0 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_4 \\ 0 \\ x_4 \end{bmatrix} = x_3 \boxed{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}} + x_4 \boxed{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}}$$

$\boxed{2.2}$

⑥ L in $\mathbb{R}^3$ is all scalar multiples of $\boxed{\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}}$.

Find orthogonal projection of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ onto L.

$$\boxed{\operatorname{proj}_L\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \frac{\overset{(*)}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}}}{\underset{(*)}{\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}}} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}} = \frac{5}{9}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 10/9 \\ 5/9 \\ 10/9 \end{bmatrix}$$

$$\operatorname{proj}_L(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{\cancel{\vec{w} \cdot \vec{w}}} \vec{w}$$

$\vec{w}$ is on line

$\vec{w}$ unit, $\vec{w} \cdot \vec{w} = 1$

$(*)$ $2+1+2 = 5$

$(*)$ $2^2 + 1^2 + 2^2 = 9$

1 T/F

5, 7, 9, 10, 16, 21

2 7, 8, 10, 43

3 7 27 41

① T/F Answers

5, 7, 9, 10, 16, 21

F, T, F, F, F, F

② 7, 8, 10, 43

F T F F

③ 7 27 41

T T T

$\downarrow$ ex

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$